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ASYMPTOTIC SOLUTION OF NATURAL CONVECTION PROBLEM IN A SQUARE CAVITY HEATED FROM BELOW

M. GRUNDMANN[†], A. MOJTABI[†] and B. VAN 'T HOF[‡]

[†]*Institut de Mécanique des Fluides, UMR CNRS/NP/UPS, 118 Route de Narbonne, 310 62 Toulouse, France*

[‡]*Department of Mathematics, University of Groningen, 9700 AV Groningen, The Netherlands*

ABSTRACT

Studies a two-dimensional natural convection in a porous, square cavity using a regular asymptotic development in powers of the Rayleigh number. Carries the approximation through to the 34th order. Analyses convergence of the resulting series for the Nusselt number in both monocellular and multicellular cases, providing insight in the validity regions of the power series.

KEY WORDS Natural convection Porous medium Square cavity Asymptotic solution Algebra code

NOMENCLATURE

- Ψ = Stream function
- T = Temperature
- Ra = Rayleigh number
- Ra_c = Critical Rayleigh number
- Nu = Nusselt number
- $\hat{Nu}$ = Modified Nusselt number
- ε = Parameter of development $\varepsilon = \sqrt{Ra - Ra_c}$
- φ, τ = Eigenfunctions
- λ = Eigenvalue
- x = Dimensionless horizontal co-ordinate
- z = Dimensionless vertical co-ordinate

INTRODUCTION

Natural convection in porous media has been the subject of many studies, the great number of publications on the subject being a testimony[1,2]. In the theoretical analysis of flow in the porous, vertical, rectangular cavity, heated from below, which is the subject of this study, Beck[3] performed a linear stability analysis of the convective solution from which he derived critical Rayleigh numbers and the type of convective flow which sets up after the first transition.

Schubert and Strauss[4] conducted a numerical study using a Galerkin method with double or triple Fourier series in the case of a square or cubic porous cavity heated from below. They proved the existence of monocellular and multicellular convective solutions for Rayleigh numbers higher than the critical values, at which the transition from conduction to convection occurs.

Caltagirone and Fabrie[5] confirmed Schubert and Strauss's results and extended their results to nonstationary flow using the same numerical approach as Schubert and Strauss. They found periodic as well as non-periodic solutions.

Riley and Winters[6] used results from bifurcation theory in their study of a porous, vertical, rectangular cavity heated from below. They showed numerically the existence and stability of

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multiple solutions and analysed the influence of the height-width cavity ratio on the stability of the solutions. Building on these results, they produced a graph of the existence and stability of stationary solutions for different Rayleigh numbers and height-width ratios.

This study was completed by Riley and Winters[7], when they analysed the Hopf bifurcations of the different solutions.

The purpose of this work is to use analytical methods for the construction of an accurate solution of the equations describing convection in a porous, vertical, rectangular cavity heated from below. The analytical solution obtained describes very well the mono-, bi- and tricellular flow in a square cavity, and can be extended to cavities of arbitrary height-width ratios.

The solution uses an asymptotic development up to 34th order in the parameter $\varepsilon = \sqrt{Ra - Ra_c}$, where Ra_c denotes the value of the critical Rayleigh number (critical for the transition from conduction to convection).

The series can be constructed analytically because every power term consists of only a finite number of two-dimensional Fourier components. To find those components and their coefficients, it is necessary to use a program for symbolic computations. In this study the Maple program was used. The results were in very good accordance with numerical results that have been computed earlier by other authors.

An estimation has been made for the convergence radius of the Nusselt number series solution, given in powers of $(Ra - Ra_c)$. The series for the monocellular solution was found to be an alternating series, with the result that the error is of the same order as the largest ignored term in the series.

The series solution that has been obtained can be used as a reference solution for the validation of future numerical codes. The discrepancy between this method and that of the Legendre spectral collocation (43×43) is less than 10^{-13} for $Ra = 45$. A 34th order development is not sufficient to obtain the radius of convergence of these series in the bi- and tricellular case. However, the computation of the Nusselt number gives results in good agreement with the numerical one.

ASYMPTOTIC APPROACH

The following system of coupled partial differential equations describes the natural convection in terms of the stream function Ψ and the temperature T:

$$\Delta\Psi = -\frac{\partial T}{\partial x} \tag{1a}$$

$$\Delta T = Ra\left(\frac{\partial\Psi}{\partial z}\frac{\partial T}{\partial x} - \frac{\partial\Psi}{\partial x}\frac{\partial T}{\partial z}\right). \tag{1b}$$

The domain Ω is the square cavity $[0, \pi] \times [0, \pi]$, heated from below, with adiabatic side walls. At the boundaries $\partial\Omega$ of the cavity, the following conditions are prescribed:

$$\Psi = 0 \qquad \forall(x,\ z) \in \partial\Omega \tag{2a}$$

$$T = 0 \qquad z = 0 \qquad \forall x \tag{2b}$$

$$T = -\pi \qquad z = \pi \qquad \forall x \tag{2c}$$

$$\frac{\partial T}{\partial x} = 0 \quad x = 0,\ \pi \qquad \forall z. \tag{2d}$$

This dimensionless formulation has been chosen in order to obtain an analytical solution in the simplest possible form.

The problem described above has a conductive solution valid for any value of Ra. It is given by

$$\Psi_0 = 0 \quad T_0 = -z. \tag{3}$$

Starting from this solution, the Rayleigh number for the solutions which set up when the Rayleigh number exceeds its critical value Ra_C can be expressed in the form

$$Ra = Ra_c + \varepsilon^2. \tag{4}$$

Let Ψ and T be written in the following asymptotic expansions

$$\Psi = \sum_{i=1}^{\infty} \varepsilon^i \psi_i \quad T = -z + \sum_{i=1}^{\infty} \varepsilon^i t_i. \tag{5}$$

Using equations (1a) and (2a), it is seen that Ψ is determined by T. Therefore Ψ can be written as

$$\Psi = \Psi(T). \tag{6}$$

One can easily see that Ψ is linear. Equations (1) and (2) can now be represented by the following symbolic form

$$D(T, \Psi(T)) = 0. \tag{7}$$

Expanding the operator D with respect to ε, the equation can be rewritten as

$$\varepsilon D'(T_0, \Psi(T_0))t_1 + O(\varepsilon^2) = 0 \tag{8}$$

with D' a linear operator. The solution t_1 is searched in a basis of eigenfunctions of the operator D'.

One can easily prove that the functions

$$\tau_{n,m} = \cos(nx)\sin(mz),\ \varphi_{n,m} = \Psi(\tau_{n,m}) = \frac{-n}{n^2+m^2}\sin(nx)\sin(mz) \tag{9}$$

are eigenfunctions of D', with eigenvalues

$$\lambda_{n,m} = \frac{Ran^2 - (n^2+m^2)^2}{n^2+m^2}. \tag{10}$$

Critical values Ra_c are those for which $\lambda_{n,m} = 0$, so we have

$$Ra_c = \frac{(n^2+m^2)^2}{n^2}. \tag{11}$$

The Nth order approximation is given by

$$\Psi_N = \sum_{i=1}^{N} \varepsilon^i \Psi(t_i) \quad T_N = -z + \sum_{i=1}^{N} \varepsilon^i t_i. \tag{12}$$

Then, an asymptotic expansion is constructed for chosen numbers m and n, which remain constant during the entire calculation. The choice $m = n = 1$ yields the monocellular solution, $n = 2$, $m = 1$ the bicellular solution, $n = 3$, $m = 1$ the tricellular, etc. Choices with $m > 1$ yield mathematical solutions which are always unstable. For chosen m and n, it is found that

$$t_1 = a_1 \tau_{n,m}. \tag{13}$$

The coefficient a_1 will be determined later.

Let T_N, $\Psi_N = \Psi\ (T_N)$ be an Nth order solution. We have

$$D(T_N, \Psi_N) = O(\varepsilon^{N+1}). \tag{14}$$

Let R_{N+1} denote the $(N+1)$th order term of the residual

$$D(T_N, \Psi_N) = \varepsilon^{N+1} R_{N+1} + O\left(\varepsilon^{N+2}\right). \tag{15}$$

Since the functions $\tau_{n,m}$ are eigenfunctions of the linear operator D', we have

$$D(T_N + \varepsilon^{N+1}\tau_{k,l}, \Psi_N + \varepsilon^{N+1}\Psi(\tau_{k,l})) = \varepsilon^{N+1}(R_{N+1} + \lambda_{k,l}\tau_{k,l}) + O\left(\varepsilon^{N+2}\right). \tag{16}$$

Therefore, let us write R_{N+1} in the form

$$R_{N+1} = \alpha_{n,m,N+1}\,\tau_{n,m} + \sum_{(k,l)\neq(n,m)} \alpha_{k,l,N+1}\,\tau_{k,l}. \tag{17}$$

Let us write t_{N+1} in the form

$$t_{N+1} = a_{N+1}\tau_{n,m} - \sum_{(k,l)\neq(n,m)} \frac{\alpha_{k,l,N+1}}{\lambda_{k,l}}. \tag{18}$$

This yields to

$$D(T_{N+1}, \Psi_{N+1}) = \varepsilon^{N+1}\,\alpha_{n,m,N+1} + O\left(\varepsilon^{N+2}\right) \tag{19}$$

with

$$\alpha_{n,m,N+1} = f_{N+1}(a_{N-1}). \tag{20}$$

For all $N > 3$, f_N is a linear function, but f_3 is cubic. The three resulting solutions yield the conductive and two convective solutions, each with cells turning in opposite directions.

Increasingly high orders of accuracy can be attained by following this method.

RESULTS

After having obtained solution for the temperature and stream function up to the 32nd order in ε, a similar expression is derived for the Nusselt number, which is a series in powers of ε^2. Therefore, the Nusselt number Nu is written as

$$Nu_N = 1 + \sum_{i=1}^{N} q_i (Ra - Ra_c)^i. \tag{21}$$

For the monocellular solution, the series is an alternating series, and the convergence radius appears to be equal to the critical Rayleigh value Ra_c. This allows us to have an improved estimation of the Nusselt number by

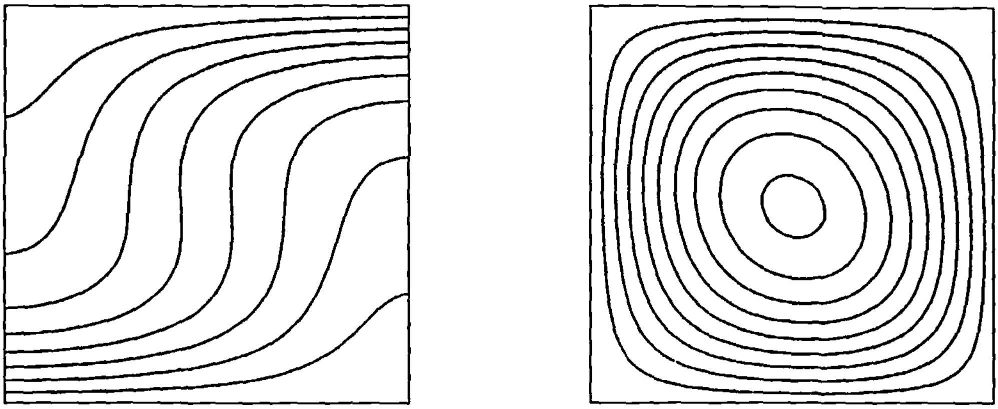

Figure 1 The isothermal lines and the stream function of the monocellular flow for $Ra = 80$

$$\hat{Nu}_N = 1 + \sum_{i=1}^{N-1} q_i (Ra - Ra_c)^i + q_N (Ra - Ra_c)^N \left(\frac{Ra_c}{Ra} \right). \tag{22}$$

This new approximation permits not only a higher accuracy, but also a better approximation of Nusselt numbers beyond the convergence radius, as shown by the comparison of the results (*Tables 1* and *3*) obtained from the asymptotic solution and those of the spectral collocation method mentioned above. A look at *Tables 2-6* concerning bi- and tricellular flow shows that the expansion to the 32nd order does not allow for an accurate estimation of the convergence radius for the power series of the Nusselt number. However, the analytical solution obtained gives a good estimation of the solution of the problem, for a wide range of Rayleigh numbers.

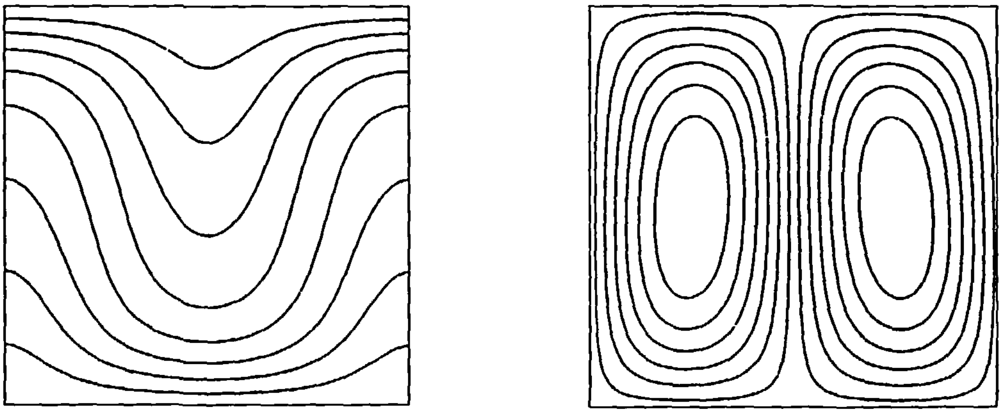

Figure 2 The isothermal lines and the stream function of the bicellular flow for $Ra = 100$

Table 1 Nusselt number for monocellular flow

Ra	Nu_N numer	Nu_N asymp.	$\hat{Nu}_N$ asymp.	$q_{16}(Ra-Ra_C)^{16}$
40	1.0261794087377	1.0261794087371	identical	$-0.11889\ 10^{-29}$
45	1.255196788925	1.255196788923	identical	$-0.29579\ 10^{-13}$
50	1.452229038043	1.452229038246	1.452229038434	$-0.89386\ 10^{-9}$
55	1.625249807210	1.625249686269	1.625249807187	$-0.44973\ 10^{-6}$
60	1.7796363848	1.7796229737	1.7796363806	$-0.39205\ 10^{-4}$
65	1.9191744666	1.9186703464	1.9191744140	$-0.12838\ 10^{-2}$
70	2.046622822	2.036822727	2.046622018	$-0.22474\ 10^{-1}$
75	2.164048659	2.043454594	2.164041026	-0.25459
80	2.273036	–	2.272986	–
85	2.374825	–	2.374574	–
90	2.470398	–	2.469441	–
95	2.560549	–	2.557751	–
100	2.645923	–	2.640312	–

Table 2 Nusselt number for bicellular flow

Ra	Nu_N numer	$\hat{Nu}_N$ asymp.	Nu_N asymp.
65	1.10641386417	1.10641386418	identical
70	1.26356799365	1.26356799368	identical
75	1.41744309151	1.41744309152	1.41744309157
80	1.568266578	1.568266573	1.568266581
85	1.71588368	1.71588338	1.71588373
90	1.85997950	1.85997217	1.85997952
95	2.0002157	2.0001116	2.0002055
100	2.1363029	2.1353021	2.1361377
105	2.2680306	2.2608550	2.2665188
110	2.3952272	2.3542884	2.3853340
115	2.5179802	2.3227308	2.4662712
120	2.636168	1.8338256	2.4111036

Table 3 Convergence radius for monocellular flow

n	q_n		
1	0.05066	19.74	19.74
2	$-0.90896\ 10^{-3}$	55.73	33.17
3	$0.21557\ 10^{-4}$	42.16	35.93
4	$-0.54908\ 10^{-6}$	39.26	36.74
5	$0.14259\ 10^{-7}$	38.51	37.08
6	$-0.36750\ 10^{-9}$	38.80	37.36
7	$0.93288\ 10^{-11}$	39.39	37.65
8	$-0.23476\ 10^{-12}$	39.74	37.90
9	$0.59111\ 10^{-14}$	39.71	38.10
10	$-0.14956\ 10^{-15}$	39.52	38.24
11	$0.37968\ 10^{-17}$	39.39	38.34
12	$-0.96396\ 10^{-19}$	39.39	38.43
13	$0.24432\ 10^{-20}$	39.46	38.51
14	$-0.61836\ 10^{-22}$	39.51	38.58
15	$0.15648\ 10^{-23}$	39.52	38.64
16	$-0.39625\ 10^{-25}$	39.49	38.69

Table 4 Convergence radius for bicellular flow

n	q_n	$\left\lvert \frac{q_{n-1}}{q_n} \right\rvert$	$\frac{1}{\sqrt[n]{\lvert q_n \rvert}}$
1	0.03242	30	30
2	$-0.1058\ 10^{-3}$	306	97
3	$0.2950\ 10^{-5}$	36	70
4	$-0.9577\ 10^{-7}$	30	57
5	$0.1431\ 10^{-8}$	67	59
6	$-0.1056\ 10^{-10}$	135	68
7	$0.1580\ 10^{-12}$	66	67
8	$-0.6737\ 10^{-14}$	24	59
9	$0.1322\ 10^{-15}$	51	58
10	$-0.6456\ 10^{-18}$	205	66
11	$-0.5572\ 10^{-20}$	116	69
12	$-0.4598\ 10^{-21}$	12	60
13	$0.1682\ 10^{-22}$	27	56
14	$-0.7496\ 10^{-25}$	224	62
15	$-0.3873\ 10^{-26}$	19	57

Table 5 Convergence radius for tricellular flow

n	q_n	$\left\lvert \frac{q_{n-1}}{q_n} \right\rvert$	$\frac{1}{\sqrt[n]{\lvert q_n \rvert}}$
1	0.01824	54	54
2	$0.1158\ 10^{-3}$	157	93
3	$0.9569\ 10^{-6}$	121	101
4	$-0.3427\ 10^{-7}$	28	73
5	$-0.1871\ 10^{-9}$	183	88
6	$0.8325\ 10^{-6}$	22	70
7	$0.1149\ 10^{-12}$	72	70
8	$-0.3298\ 10^{-14}$	34	64
9	$-0.5794\ 10^{-16}$	57	64
10	$0.1234\ 10^{-17}$	43	61
11	$0.3220\ 10^{-19}$	41	59
12	$-0.5826\ 10^{-21}$	55	59
13	$-0.1831\ 10^{-22}$	32	56
14	$0.2508\ 10^{-24}$	73	57

Table 6 Nusselt number for tricellular flow

Ra	Nu_N numer.	Nu_N asymp.
110	1.0617268324	1.00617268331
120	1.201570134198	1.20157013429
130	1.421001419	1.421001137
140	1.659415	1.659337
150	1.9093	1.9058
160	2.1639	2.1067
170	2.4177	1.9743

CONCLUSIONS

Recent developments in the field of symbolic computation have made possible the computation of very accurate analytical solutions of non-linear problems. We have obtained results for the problem of natural convection in porous media in a rectangular cavity for a range of Rayleigh numbers exceeding three times the critical value for the first transition. In the case of a monocellular flow, the alternating series is found to be convergent for $Ra \leq 8\pi^2$, a value very close to the critical $Ra_c = 81.01$, found by Riley and Winters[6], above which bicellular flow becomes stable. The error in the determination of the Nusselt number has been evaluated very carefully for several values of Ra. The solution thus obtained could serve as a reference solution for the validation of numerical codes.

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